φ and the Fibonacchi sequence

The Fibonacci sequence, $(0, 1, 1, 2, 3, 5, 8, 13, 21, \dots)$ is calculated by adding two consecutive elements to get the next one, that is

$$a_n = a_{n-1} + a_{n-2}, \quad \text{starting with } a_0 = 0 \text{ and } a_1 = 1$$

(We denote the nth element of a sequence by a_n, starting with the zeroth.)

It is a well-known fact in mathematics that sequences like that (where elements are calculated by adding and substracting multiples of previous elements) can be written as exponential functions. That means, there is a number x (in fact in this case there are two) such that $a_n = x^n$ satisfies $a_n = a_{n-1} + a_{n-2}$. We now will calculate these values x.

Assuming, as stated, $a_n = x^n$, we put this into the Fibonacci equation and get

$$x^n = x^{n-1} + x^{n-2}.$$

We divide the equation by x^{n-2} and substract $x + 1$ which gives us

$$x^2 - x - 1 = 0$$

This is a quadratic equation. We all learned how to solve those in school by using the well-known quadratic formula. The result is two different solutions, namely

$$x_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad x_2 = \frac{1-\sqrt{5}}{2}.$$

The first one, x_1 is just the *Golden Section*, φ. What about x_2?

Let's try multiplying them: $x_1 \cdot x_2 = \frac{1+\sqrt{5}}{2}\frac{1-\sqrt{5}}{2} = \frac{1-5}{4} = -1$. This means

$$x_2 = -\frac{1}{x_1} = -x_1^{-1} = -\varphi^{-1}.$$

The sequences $a_n = x_1^n$ and $a_n = x_2^n$ both satisfy the Fibonacci equation, but not the start values ($a_0 = 0$ and $a_1 = 1$).

If x_1^n and x_2^n satisfy the Fibonacci equation, so do $\alpha \cdot x_1^n$ and $\beta \cdot x_2^n$ (for any real numbers α and β) and also $\alpha \cdot x_1^n + \beta \cdot x_2^n$. If you doubt that fact, just put $a_n = \alpha \cdot x_1^n + \beta \cdot x_2^n$ into the Fibonacci equation and do the calculation for yourself.

So the task now is to find two numbers α and β such that the resulting sequence a_n satisfies $a_0 = 0$ and $a_1 = 1$ too. Therefore we need

$$a_0 = \alpha \cdot x_1^0 + \beta \cdot x_2^0 = \alpha + \beta = 0$$

$$a_1 = \alpha \cdot x_1^1 + \beta \cdot x_2^1 = \alpha\frac{1+\sqrt{5}}{2} + \beta\frac{1-\sqrt{5}}{2} = 1$$

From the first equation we get $\beta = -\alpha$ which we put into the second one to get

$$\alpha \frac{1+\sqrt{5}}{2} - \alpha \frac{1-\sqrt{5}}{2} = \alpha\Big(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\Big) = \alpha \cdot \sqrt{5} = 1$$

and therefore $\alpha = 1/\sqrt{5}$ (and $\beta = -1/\sqrt{5}$).

Now we can put everything together and see

$$\boxed{a_n = \frac{1}{\sqrt{5}}\big(\varphi^n - (-\varphi)^{-n}\big) \quad \text{is the Fibonacci sequence.}}$$

Now let's assume that n becomes very large. Then $(-\varphi)^{-n} \approx (-0.618)^n$ becomes very near 0 and therefore $a_n \approx \frac{1}{\sqrt{5}}\varphi^n$.

If we now calculate the ratio of two consecutive Fibonacci numbers we see

$$\frac{a_{n+1}}{a_n} \approx \frac{\frac{1}{\sqrt{5}}\varphi^{n+1}}{\frac{1}{\sqrt{5}}\varphi^n} = \varphi.$$

So we see that the ratio nears φ, the Golden Section.